Some Considerations on Lattice Gauge Fixing

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Abstract

Some problems related to Gribov copies in lattice gauge-fixing and their possible solution are discussed.

1 Gribov and Gauge-Fixing Problem

The Faddeev-Popov[1] quantization gives a meaning to the formal (euclidean) expectation value of a gauge invariant observable operator:

$$\langle O \rangle = \frac{\int \delta A \exp[-S(A)]\; O(A)}{\int \delta A \exp[-S(A)]} \tag{1}$$

The Faddeev-Popov method requires the choice of a gauge fixing condition:

$$f(A) = 0 \tag{2}$$

in terms of which we define $\Delta(A)$ as:

$$\Delta(A) \cdot \int D\Omega \delta[f(A^{\Omega})] = 1 \tag{3}$$

In eq.(3), $D\Omega$ denotes the invariant measure on the gauge group G. It is easy to show that $\Delta(A)$ is gauge invariant:

$$\Delta(A^\Omega) = \Delta(A) \tag{4}$$

We then get the following gauge-fixed expression for $\langle O \rangle$:

$$\langle O \rangle = \frac{\int \delta A \exp[-S(A)] \Delta(A) \delta[f(A)] O(A)}{\int \delta A \exp[-S(A)] \Delta(A) \delta[f(A)]} \tag{5}$$

Let us consider, at the moment, the academic situation in which Gribov copies[2] are absent. Then, if A satisfies $f(A) = 0$, we get an explicit expression for $\Delta(A)$:

$$\Delta(A) = \det \left. \frac{\delta f(A^\Omega)}{\delta \Omega} \right|_{\Omega=1} \tag{6}$$

Choosing, e.g., $f(A) = \partial_\mu A^\mu$ as the gauge fixing condition, we have:

$$\langle O \rangle = \frac{\int \delta A \exp[-S(A)] \det[\partial D(A)] \delta[\partial_\mu A^\mu] O(A)}{\int \delta A \exp[-S(A)] \det[\partial D(A)] \delta[\partial_\mu A^\mu]} \tag{7}$$

More generally we can eliminate the δ-function from the functional integrand as:

$$\langle O \rangle = \frac{\int \delta A e^{[-S(A) - \frac{1}{2\alpha} \int (\partial A)^2]} \det[\partial D(A)] O(A)}{\int \delta A e^{[-S(A) - \frac{1}{2\alpha} \int (\partial A)^2]} \det[\partial D(A)]} \tag{8}$$

Eq.(8) can be linearized through the introduction of the Lagrange multipliers $\lambda(x)$:

$$\langle O \rangle = \frac{\int \delta A \delta \lambda e^{[-S(A) + i \int \lambda \partial A - \frac{\alpha}{2} \int \lambda^2]} \det[\partial D(A)]) O(A)}{\int \delta A \delta \lambda e^{[-S(A) + i \int \lambda \partial A - \frac{\alpha}{2} \int \lambda^2]} \det[\partial D(A)])} \tag{9}$$

It is now possible to rewrite eqs.(8),(9) introducing ghost and antighost fields, $c(x)$ and $\bar{c}(x)$:

$$\langle O \rangle = \frac{\int \delta A \delta c \delta \bar{c} e^{[-S(A) - \frac{1}{2\alpha} \int (\partial A)^2 - \int \partial \bar{c} D(A) c]} O(A)}{\int \delta A \delta c \delta \bar{c} e^{[-S(A) - \frac{1}{2\alpha} \int (\partial A)^2 - \int \partial \bar{c} D(A) c]}} = \tag{10}$$

$$= \frac{\int \delta A \delta \lambda \delta c \delta \bar{c} e^{[-S(A) + i \int \lambda \partial A - \frac{\alpha}{2} \int \lambda^2 - \int \partial \bar{c} D(A) c]} O(A)}{\int \delta A \delta \lambda \delta c \delta \bar{c} e^{[-S(A) + i \int \lambda \partial A - \frac{\alpha}{2} \int \lambda^2] - \int \partial \bar{c} D(A) c]}} \tag{11}$$

In the formulation given in eq.(11), the theory admits a nilpotent ($\delta^2 = 0$) BRST symmetry[3]:

$$\begin{aligned} \delta A_\mu &= D_\mu(A) c \\ \delta \bar{c} &= i\lambda \\ \delta c &= \frac{1}{2} c c \\ \delta \lambda &= 0 \end{aligned} \tag{12}$$

BRST invariance follows from the fact that eq.(11) can be rewritten, through eqs.(12), as:

$$\langle O \rangle = \frac{1}{Z'} \int D\mu e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{\delta \int f \bar{c}} O(U) \tag{13}$$

where:

$$Z' = \int D\mu e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{\delta \int f \bar{c}} \tag{14}$$

and:

$$D\mu \equiv DA d\lambda d\bar{c} dc \tag{15}$$

is the BRST-invariant measure.

Gribov copies correspond to multiple Ω solutions of the equation

$$f(A^\Omega) = 0 \tag{16}$$

for a given a gauge field configuration $A^a_\mu(x)$. Labeling the different solutions of eq.(16) by Ω_i, we get from eq.(3):

$$\Delta(A)^{-1} = \sum_i \frac{1}{\left| \det \frac{\delta f(A^\Omega)}{\delta \Omega} \right|_{\Omega_i}} \tag{17}$$

Although correct, the use of eq.(17) is very inconvenient: in this way the ghost formulation is lost and, with it, the related BRST invariance.

An alternative procedure which maintains BRST symmetry also in presence of Gribov copies, has been suggested long ago[4]. It consists in the observation that the quantity:

$$n(A) \equiv \int D\Omega \det \left[\frac{\delta f(A^\Omega)}{\delta \Omega} \right] \delta[f(A^\Omega)] \tag{18}$$

is a sum of ± 1, which counts the intersections (with sign) of the gauge orbit with the gauge fixing surface. As a consequence of an index theorem, it turns out that $n(A)$ is independent of A_μ. If we could show, in addition, that $n(A) \neq 0$, the Faddeev-Popov formulation and the BRST symmetry would follow at once. In particular, if the gauge condition $f(A) = \partial_\mu A^\mu$ were chosen, we would get precisely eqs.(10),(11) even in presence of Gribov copies.

Lattice regularization[5] offers a unique opportunity to study this problem, since, due to the compactness of the lattice gauge fields, U_μ, both the gauge fixed and non gauge fixed versions of the path integral are meaningful. However, as explained in section 2, precisely because of compactness, we can show[6] that $n(A) = 0$.

2 Neuberger problem ($n(A) = 0$)

We start with the lattice regularized expectation value of a gauge invariant operator $O(U)$:

$$\langle O \rangle = \frac{1}{Z} \int DU e^{-S(U)} O(U) \tag{19}$$

where:

$$Z = \int DU e^{-S(U)} \tag{20}$$

It is easy to introduce a gauge-fixing $f(A)$ along the same lines discussed in section 2:

$$\langle O \rangle = \frac{1}{Z'} \int D\mu e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{\delta \int f \bar{c}} O(U) \tag{21}$$

where:

$$Z' = \int D\mu e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{\delta \int f \bar{c}} \tag{22}$$

and:

$$D\mu \equiv DU d\lambda d\bar{c} dc \tag{23}$$

In eqs.(21),(22), δ denotes a nilpotent ($\delta^2 = 0$) Lattice BRST transformation defined by:

$$\delta U_\mu = c(x) U_\mu(x) - U_\mu(x) c(x + a\hat{\mu}) \tag{24}$$

$$\delta \bar{c} = i\lambda \tag{25}$$

$$\delta c = \frac{1}{2} c c \tag{26}$$

$$\delta \lambda = 0 \tag{27}$$

Following [6], we can now define:

$$F_O(t) \equiv \int D\mu e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{t\delta \int f\,\bar{c}} O(U) \tag{28}$$

so that, due to nilpotency:

$$\frac{dF_O(t)}{dt} \equiv \int D\mu [\delta \int f\,\bar{c}] e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{t\delta \int f\,\bar{c}} O(U) = \tag{29}$$

$$= \int D\mu \delta [\int f\,\bar{c}\; e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{t\delta \int f\,\bar{c}} O(U)] = 0 \tag{30}$$

On the other hand we also have:

$$F_O(0) = \int DU d\lambda d\bar{c} dc\; e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} O(U) = 0 \tag{31}$$

as a consequence of Berezin integration rules, since the integrand of eq.(31) does not contain ghost, nor antighost fields.

We get therefore:

$$F_O(1) = \int D\mu e^{-S(U)} e^{-\frac{\alpha}{2}\int \lambda^2} e^{\delta \int f\,\bar{c}} O(U) = 0 \tag{32}$$

and the expectation value of any observable assumes the form $\langle O \rangle = \frac{0}{0}$.

As discussed in section 3, this situation is the consequence of a cancellation among Lattice Gribov copies.

3 Toy Abelian Model

In this section we will consider a zero dimensional prototype of abelian BRST symmetry with compact variables[7] which will clarify the nature of the problem and a possible way out. The model consists of one "link" variable U, which we choose to parametrize through its phase, as:

$$U = e^{iaA} \tag{33}$$

where:
$$-\frac{\pi}{a} \leq A \leq \frac{\pi}{a} \tag{34}$$
a is a parameter, reminiscent of the lattice spacing in more realistic situations, whose limit $a \to 0$ will be used to connect the periodic, compact case to the non compact one.

We define:
$$N \equiv \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \tag{35}$$

The gauge-fixed version of the "functional" integral in eq.(35), is:
$$N' = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{+\infty} d\lambda \int d\bar{c} dc e^{-\frac{\alpha}{2}\lambda^2} e^{\delta[\bar{c} f(A)]} \tag{36}$$
where δ denotes the (nilpotent $\delta^2 = 0$) BRST-like transformation:
$$\begin{aligned} \delta A &= c \\ \delta c &= 0 \\ \delta \bar{c} &= i\lambda \\ \delta \lambda &= 0 \end{aligned} \tag{37}$$
Going through the same steps as in section 2, we conclude that N' suffers from the Neuberger disease:
$$N' = 0 \tag{38}$$
This can also be checked through an explicit calculation:
$$\begin{aligned} N' &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{+\infty} d\lambda \int d\bar{c} dc e^{-\frac{\alpha}{2}\lambda^2} e^{i\lambda f(A)} e^{-\bar{c} f'(A) c} = \\ &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{+\infty} d\lambda \, e^{-\frac{\alpha}{2}\lambda^2} e^{i\lambda f(A)} f'(A) = \\ &= \sqrt{\frac{2\pi}{\alpha}} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} df(A) e^{-\frac{f(A)^2}{2\alpha}} = 0 \end{aligned} \tag{39}$$

for a periodic, non-singular, $f(A)$. The reason why we need a periodic $f(A)$ is that we want BRST Identities to be satisfied. This is crucial to show independence on α of gauge-invariant observables. The prototype of BRST Identities is:

$$\langle \delta\Gamma \rangle = 0 \tag{40}$$

where Γ is any quantity with ghost number -1. If we choose:

$$\Gamma \equiv \bar{c} F(A, \lambda) \tag{41}$$

so that:

$$\delta\Gamma \equiv \delta[\bar{c} F(A,\lambda)] = i\lambda F(A,\lambda) - \bar{c} F'(A,\lambda) c \tag{42}$$

where the $'$ denotes the derivative with respect to A, we have:

$$\begin{aligned} \langle \delta\Gamma \rangle &\equiv \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{+\infty} d\lambda \int d\bar{c} dc e^{-\frac{\alpha}{2}\lambda^2} e^{\delta[\bar{c} f(A)]} \delta\Gamma = \\ &= \int_{-\infty}^{+\infty} d\lambda e^{-\frac{\alpha}{2}\lambda^2} e^{i\lambda f(A)]} F(A,\lambda) \Bigg|_{A=-\frac{\pi}{a}}^{A=\frac{\pi}{a}} = 0 \end{aligned} \tag{43}$$

which can only be satisfied for a periodic (in A) gauge fixing condition, $f(A)$, and $F(A, \lambda)$. In particular, for $\alpha = 0$, eq.(36) becomes:

$$\begin{aligned} N' = \lim_{\alpha \to 0} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{+\infty} d\lambda\, e^{-\frac{\alpha}{2}\lambda^2} e^{i\lambda f(A)} f'(A) = \\ = 2\pi \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA\, f'(A)\, \delta\left(f(A)\right) = 0 \end{aligned} \tag{44}$$

which displays the Gribov nature of the paradox: a periodic $f(A)$ has an even number of zeroes which contribute alternatively ± 1 to eq.(44) and cancel exactly.

Within this toy abelian model, the solution of the "Gribov problem" is simple. It is enough to substitute the gauge fixing δ-function with a periodic δ-function[8]:

$$\delta \Rightarrow \delta_P \tag{45}$$

with:

$$\delta_P(x) \equiv \sum_{n=-\infty}^{+\infty} \delta(x - n\frac{2\pi}{a}) = \frac{a}{2\pi} \sum_{n=-\infty}^{+\infty} e^{inax} \equiv \frac{a}{2\pi} \sum_{n=-\infty}^{+\infty} e^{i\lambda_n x} \tag{46}$$

In eq.(46) we put:

$$\lambda_n \equiv na \tag{47}$$

Extending the formulation with the inclusion of a λ_n^2 at the exponent, analogous to eq.(36), we then have:

$$N' \Rightarrow N'' \tag{48}$$

where:

$$N'' = a \sum_{n=-\infty}^{+\infty} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA e^{-\frac{\alpha}{2}\lambda_n^2} e^{i\lambda_n f(A)} f'(A) \tag{49}$$

This formulation admits an obvious BRST invariance under transformations similar to eq.(37), provided we interpret the variation of the antighost as:

$$\delta\bar{c} = i\lambda_n \tag{50}$$

We have, in analogy with eq.(36):

$$N'' = a \sum_{n=-\infty}^{+\infty} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int d\bar{c} dc e^{-\frac{\alpha}{2}\lambda_n^2} e^{\delta[\bar{c} f(A)]} \tag{51}$$

Invariance under these modified BRST transformations is enough for all purposes related to gauge invariance.

The advantage of having a discretize set of Lagrange multipliers λ_n, eq.(47), is that we are now free to chose a non-periodic "gauge fixing" condition $f(A)$ such that:

$$f(A + \frac{2\pi}{a}) = f(A) + \frac{2\pi}{a} \tag{52}$$

still respecting BRST Identities, eq.(40). In fact, while the integrand of eq.(49) is still periodic, the condition stated in eq.(52) evades the cancellation

among Gribov copies because $f(A)$ has an odd number of zeroes. Another way of stating this fact is to recognise that $\exp(it\lambda_n f(A))$ is only periodic for integer t's and Neuberger's argument, which requires taking a derivative with respect to t, is avoided. When $a \to 0$ we recover the continuum BRST formulation in analogy to the way in which we get the Fourier integral from the Fourier series:

$$\lim_{a\to 0} N'' = \lim_{a\to 0} a \sum_{n=-\infty}^{+\infty} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA e^{-\frac{\alpha}{2}\lambda_n^2} e^{i\lambda_n f(A)} f'(A) =$$

$$= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{+\infty} d\lambda\, e^{-\frac{\alpha}{2}\lambda^2} e^{i\lambda f(A)} f'(A) \tag{53}$$

4 $U(1)$ Gauge Theory

The case of the $U(1)$ Gauge Theory can be immediately treated along the lines of the toy model. We parametrize the gauge field $U_\mu(x)$ as:

$$U_\mu(x) \equiv e^{iaA_\mu(x)} \tag{54}$$

The BRST variation of $A_\mu(x)$, induced by eq.(24) is:

$$\delta A_\mu(x) = \frac{c(x+a\hat{\mu}) - c(x)}{a} \tag{55}$$

and we can chose, for example, a discretization of the Lorentz gauge-fixing:

$$f(A) = \sum_\mu \frac{[A_\mu(x+a\hat{\mu}) - A_\mu(x)]}{a} \tag{56}$$

In this case:

$$\int \delta A \delta c \delta \bar{c} \sum_{n(x)} e^{[-S(A)+ia^4 \sum_x \lambda_n(x) \sum_\mu \frac{[A_\mu(x+a\hat{\mu})-A_\mu(x)]}{a} - \frac{\alpha}{2} a^4 \sum_x \lambda_n^2(x) - \int \partial\bar{c}\partial c]} \tag{57}$$

where:

$$\lambda_n(x) = \frac{n(x)}{a^2} \tag{58}$$

5 Conclusions

The Fujikawa-Hirschfeld-Sharpe proposal[4] seems to be viable, at least in the abelian compact case. More work is needed to clarify the considerably more difficult case of non abelian compact gauge fields.

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